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OPTIMAL CONTROL IN DETERMINISTIC INVENTORY MODELS

by

Alan W. McMasters

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ABSTRACT:

The use of Pontryagin's Maximum Principle in analyses of deterministic inventory models requires less restrictive assumptions about the nature of order policies than can be allowed when only the differential calculus is used. Two analyses are presented to illustrate this fact. The first involves a periodic review model with no shortages allowed. The production and holding cost functions were kept as general as possible. The second analysis involves a periodic review model with shortages allowed and linear production and inventory costs. As would be expected, the optimal order policies obtained are more general than those obtained in the past.

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I. INTRODUCTION

This paper is devoted to deterministic inventory models for three primary reasons. The first is that such models provide considerable insight about inventory models which may not be deterministic. The second is that many stochastic models are approximated by deterministic ones (for example, by using expected demand rates in place of deterministic ones). The third is the limited variety of deterministic models which have been studied in the literature. The first two reasons are of particular importance to people who are applying inventory theory to real-world inventory systems.

Past work [3] on deterministic inventory models, with few exceptions, has assumed the nature of the ordering policy to within one or two unknown parameters. For example, a periodic review model might specify that a quantity Q can only be ordered at the end of a known period of time T. The unusual problem would be to determine the value of the parameter Q which minimizes the total purchase or production plus holding costs. If, however, the nature of the optimal policy need not be specified prior to the analysis then we would expect that the complete nature of the optimal policy will result.

Unfortunately, if the nature of the ordering policy is not specified then considerable complexity of analysis can result if one attempts to use the differential calculus (see, for example, the work of Arrow and Karlin [2]). This complexity can be reduced however if the Maximum Principle of Pontryagin et al [4] is used in determining the optimal ordering policy. Such an approach will be taken in this paper.

The paper begins with an analysis of the no-stockout periodic review problem posed by Arrow and Karlin [2] to illustrate the approach and the

simplicity of analysis that it provides. Then an analysis of a periodic review problem with stockouts allowed is made to show how the allowing of stockouts complicates the optimal control problem.

II. NO-STOCKOUTS PROBLEM

Model Formulation--This is a generalization of the problem formulated by Arrow and Karlin [2]. We are interested in scheduling the production of an item so as to minimize its production plus holding costs over a specified period of time T. We will assume a zero lead time. Let

$I(t)$ = the inventory level at time t;

$r(t)$ = demand rate at time t;

$u(t)$ = the production rate at time t.

Both $r(t)$ and $u(t)$ are to be non-negative.

All demands are assumed to be met and, as a consequence,

$$I(t) = I(0) + \int_0^t [u(\tau) - r(\tau)]d\tau \geq 0 \qquad (1)$$

where $I(0)$ represents the on-hand inventory at the beginning of the time period. We can rewrite (1) as

$$\int_0^t u(\tau)d\tau \geq \int_0^t r(\tau)d\tau - I(0) \qquad (2)$$

and consider it to represent a constraint on $u(t)$.

The production and holding costs per unit time will be represented by $c(u(t))$ and $h(I(t))$ respectively. We will assume that they are monotone-increasing and continuously differentiable in their respective arguments. No other costs are assumed to exist. The total production plus holding costs for the period are given by (3).

$$J(u) = \int_0^T [c(u(t)) + h(I(t))]dt \quad . \qquad (3)$$

Our objective is to find a policy $u(t) \geq 0$ which satisfies (2) and minimizes (3).

The Maximum Principle for the Problem--For convenience, we will refer to (3) in the following form in our analysis:

$$J(u) = \int_0^T f_0(u,I)dt \quad .$$

In applying the Maximum Principle a function called the Hamiltonian is created as follows:

$$H(u,I,p_0,p_1,t) = p_0(t)f_0(u,I) + p_1(t)\frac{dI}{dt} \tag{4}$$

The derivative $\frac{dI}{dt}$ is obtained by differentiating (1). The Hamiltonian, upon introducing the expressions for f_0 and $\frac{dI}{dt}$, is given by (5) where t has been suppressed.

$$H(u,I,p_0,p_1) = p_0[c(u)+h(I)] + p_1[u-r] \tag{5}$$

For fixed I, p_0, and p_1 let $M(I,p_0,p_1) = \max H(u,I,p_0,p_1)$ over the domain of $u(t)$ values satisfying $u(t) \geq 0$ and $I(t) \geq 0$.

The necessary conditions for optimality of our problem from the Maximum Principle are given by the following theorem [4].

Theorem: An admissible order policy $u^*(t)$ and the resulting inventory level $I^*(t)$ are optimal if there exist continuous functions $p_0(t)$ and $p_1(t)$ which satisfy

$$\frac{dI}{dt} = \frac{\partial H}{\partial p_1} \quad ; \tag{6}$$

$$\frac{dp_0}{dt} = 0 \quad ; \tag{7}$$

$$\frac{dp_1}{dt} = -\frac{dH}{dI} \quad ; \tag{8}$$

$$p_0(T) = -1 \quad ; \tag{9}$$

$$p_1(T) = 0 \quad ; \tag{10}$$

such that

$$H(u^*, I^*, p_0, p_1) = M(I^*, p_0, p_1)$$

for all t in $[0,T]$.

The condition given by (10) is a consequence of the transversality condition for a free right-hand end condition for $I(t)$; that is, we are not going to force $I(T)$ to take on any specified value. This is a point of major difference between this analysis and that of Arrow and Karlin[2] and Adiri and Ben-Israel [1]. In both of these studies the assumption was made that $I^*(T) = 0$. If, however, $I(T) = 0$ is really optimal then the results of the analysis should show it. We shall see below that they do.

<u>The Optimal Policy</u>--Examining the conditions from the theorem we get back from (6) our constraint that $\frac{dI}{dt} = u(t) - r(t)$ and consequently that

$$I(t) = I(0) + \int_0^t [u(\tau)-r(\tau)]d\tau$$

with $I(0)$ as the given initial condition. From (7) and (9) we get $p_0(t) = -1$ for all t in $[0,T]$. From (8) we get

$$\frac{dp_1}{dt} = -p_0 \frac{d\ h(I)}{dI}$$

and, because $p_0(t) = -1$,

$$\frac{dp_1}{dt} = \frac{d\ h(I)}{dt} \quad .$$

Solving for $p_1(t)$,

$$p_1(t) = \int_0^t \frac{d\,h(I)}{dI}\,d\tau + p_1(0) \quad . \tag{11}$$

Because of (10) $p_1(T) = 0$ is needed and it follows that

$$p_1(0) = -\int_0^T \frac{d\,h(I)}{dI}\,d\tau$$

and, therefore, (11) reduces to

$$p_1(t) = -\int_t^T \frac{d\,h(I)}{dI}\,d\tau \quad . \tag{12}$$

We will find $\frac{dH}{du}$ helpful in selecting $u^*(t)$. From (5) we obtain

$$\frac{dH}{du} = p_0 \frac{d\,c(u)}{du} + p_1 \tag{13}$$

Substitution of $p_0(t) = -1$ and $p_1(t)$ as given by (12) into (13) then results in

$$\frac{dH}{du} = -\frac{d\,c(u)}{du} - \int_t^T \frac{d\,h(I)}{dI}\,d\tau \quad . \tag{14}$$

Because we assumed $\frac{dc}{du} > 0$ and $\frac{dh(I)}{dI} > 0$ it is immediately evident from (14) that $\frac{dH}{du} < 0$ for t in $[0,T]$ and therefore that the maximum value of H occurs for the smallest value of $u(t)$ which satisfies $u(t) \geq 0$ and $I(t) \geq 0$. Therefore, we deduce that

$$u^*(t) = \begin{cases} 0 & \text{for } 0 \leq t < t_1 \ ; \\ r(t) & \text{for } t_1 \leq t \leq T \ , \end{cases} \tag{15}$$

where t_1 is obtained (1) when $I(t_1) = 0$; that is, from

$$\int_0^{t_1} r(\tau)d\tau = I(0) \quad .$$

In other words, the optimal order policy is to first use up the on-hand inventory, then order to just meet current demand. We see that no inventory is held after the initial inventory $I(0)$ is consumed. It is therefore immediately evident that $I(T) = 0$ under this policy.

Comments--The optimal policy is intuitively appealing. When there are costs associated with both production and holding inventory we would like to avoid incurring either if we can. The constraints require, however, that we meet demand when it occurs. Waiting to produce until we have used up the existing inventory $I(0)$ keeps inventory holding costs to a minimum because we reduce the inventory as fast as possible while incurring no production costs. Producing only to meet demand after the inventory $I(0)$ is used up insures that $I(t)$ is zero and therefore that no holding costs are incurred. The production costs incurred are only those necessary to meet the demand. The only restrictions we placed on $c(u)$ and $h(I)$ were that

$$\frac{dc(u)}{du} > 0 \quad \text{for} \quad u \geq 0 \quad ,$$

$$\frac{dh(I)}{dI} > 0 \quad \text{for} \quad I \geq 0 \quad .$$

We could also allow $\frac{dh(I)}{dI} \geq 0$ and get the same result. For example, if there is no holding cost for items in storage we would get (14) in the reduced form

$$\frac{dH}{du} = -\frac{dc}{du}$$

and the optimal policy given by (15) still holds. Our result is much more general than that of Arrow and Karlin [2] who required $c(u)$ to be convex and continuously twice differentiable in addition to being monotone increasing. Further they only considered the case of $h(I) = hI$. Our result is also more general than that of Adiri and Ben-Israel [1] even though they used a general $h(I)$.

III. STOCKOUTS-ALLOWED PROBLEM

Model Formulation -- The problem when stockouts are allowed in general form is to find $u(t) \geq 0$ which minimizes (3); ie,

$$J(u) = \int_0^T [c(u(t)) + h(I(t))]\, dt \quad ,$$

where now $h(I)$ represents holding and shortage costs.

The Hamiltonian and $\frac{dH}{du}$ in this case are the same as (5) and (14) respectively. They are repeated here for convenience.

$$H(u,I,p_0, p_1) = p_0\ [c(u) + h(I)] + p_1\ [u-r] \quad .$$

$$\frac{dh}{du} = -\frac{dc(u)}{du} - \int_t^T \frac{dh(I)}{dI}\, d\tau \quad .$$

The theorem stated in the previous section still provides the necessary conditions for optimal $u(t)$. Although the assumption that $\frac{dc(u)}{du} > 0$ for $u \geq 0$ remains reasonable, the assumption $\frac{dh}{dI} \geq 0$ is no longer reasonable over all possible $I(t)$ values because the "stockouts-allowed" case can result in $I(t) < 0$. We expect instead that $\frac{dh}{dI} \leq 0$ for $I(t) < 0$ because a reduction in the number of shortages should result in a decrease in stockout costs. It is therefore possible to get

$$\int_t^T \frac{dh(I)}{dI}\, dr < 0 \quad ,$$

and if this integral is large enough it could cause

$$\frac{dH}{du} \geq 0$$

for some values of t in $[0,T]$. If this should happen then optimal u would be infinite unless we impose an upper bound on its range. We will therefore assume $u(t) \le b < \infty$.

It is important to observe that we will always have

$$\left.\frac{dH}{du}\right|_{t=T} = -\frac{dc(u)}{du} > 0 .$$

We can therefore say that regardless of the form of $h(I)$ the optimal ordering policy will always have $u^*(T) = 0$. This information will be extremely useful in the analysis of the problem allowing stockouts.

To progress beyond this observation, we must select a form for $h(I)$. Suppose we assume a linear form as follows:

$$h(I(t)) = \begin{cases} -h_1 I(t) & \text{for} \quad I(t) \le 0 \\ h_2 I(t) & \text{for} \quad I(t) \ge 0 . \end{cases}$$

where $h_1 > 0$ and $h_2 > 0$ represent the shortage and holding costs per unit of inventory per unit time. When we attempt to obtain the derivative $\frac{dh}{dI}$ of this form we realize that it does not exist at $I(t) = 0$. If, instead, we assume that $h(I(t))$ smoothly transitions from $-h_1 I(t)$ at $I(t) = -\varepsilon$ to $+h_2 I(t)$ at $I(t) = +\varepsilon$ for $\varepsilon > 0$ but very small then the derivative will exist at any point in the ε-neighborhood of $I(t) = 0$. If we assume further that the minimum point of the $h(I)$ curve occurs at $I = 0$ we can approximate $\frac{dh}{dI}$ by the following form:

$$\frac{dh}{dI} = \begin{cases} -h_1 & \text{for} \quad I(t) < 0 \\ 0 & \text{for} \quad I(t) = 0 \\ h_2 & \text{for} \quad I(t) > 0 . \end{cases}$$

For convenience we will also assume $\frac{dc}{du} = k$ and $r(t) = r$ where k and r are positive constants.

Preliminary Observations about the Optimal Policies--Under the model assumptions we see that if $I(t) > 0$ over $[0,T]$ then

$$\frac{dH}{du} = -k - \int_t^T h_2 d\tau = -k - h_2[T-t] \tag{16}$$

and we realize that $\frac{dH}{du} < 0$ for all t in $[0,T]$. The optimal order policy is $u^*(t) = 0$. Now if $I(0) \geq rT$, we will get $I(t) = I(0) - rt > 0$ for $u(t) = 0$ except possibly at T where $I(T) = 0$ when $I(0) = rT$.

Suppose next that $I(0) < 0$ and we order nothing, then $I(t) = I(0) - rt < 0$ for all t in $[0,T]$. In this case,

$$\frac{dH}{du} = -k + h_1 [T-t] \quad , \tag{17}$$

and it is possible to get $\frac{dH}{du} \geq 0$. If we set $\frac{dH}{du} = 0$ and solve for t in (17) we get

$$\hat{t} = T - \frac{k}{h_1} \quad . \tag{18}$$

In the range $t < \hat{t}$ we have $\frac{dH}{du} > 0$; in the range $\hat{t} < t \leq T$ we have $\frac{dH}{du} < 0$.

Because $T, k,$ and h_1 are parameters of the problem it is possible to have $\hat{t} < 0$ so that $\frac{dH}{du} < 0$ over all t in $[0,T]$. This will occur when $T < \frac{k}{h_1}$. The optimal policy when this happens is to order nothing; ie, $u^*(t) = 0$.

When $T > \frac{k}{h_1}$ then $\frac{dH}{du} > 0$ if we order nothing. However, $\frac{dH}{du} > 0$ implies that $u^*(t) = b$. If $b > r$ then it is possible to get $I(t) > 0$

towards the end of the time period and (17) would no longer apply. Some form such as (19) is suggested for t in $[0,t_1]$ where t_1 is the value of t when $I(t) = 0$.

$$\frac{dH}{du} = - k + h_1[t_1 - t] - h_2[T-t_1] \quad . \tag{19}$$

In spite of the additional complex forms for $\frac{dH}{du}$ which may arise, they will all have an appearance similar to (19).

<u>The Optimal Policies</u>--The optimal policies for the stockouts-allowed problem will be stated before the arguments leading to these policies are presented. Before stating the optimal policies it is convenient to define several quantities.

Let

$$I_1 = \frac{h_1[r-b]\hat{t}}{h_1+h_2} \quad , \tag{20}$$

$$I_2 = [r-b]\hat{t} \quad , \tag{21}$$

$$I_3 = r\hat{t} \quad , \tag{22}$$

$$\hat{\hat{t}} = \frac{[h_1+h_2]I(0) - h_1[r-b]\hat{t}}{h_1b+h_2r} \quad , \tag{23}$$

where $\hat{t}$, given by (18), is repeated here for convenience,

$$\hat{t} = T - \frac{k}{h_1} \quad .$$

The optimal order policies are:

1. If $\hat{t} > 0$ and (a) $0 < b < r$, $I(0) \leq I_1$, or (b) $b \geq r$, $I(0) \leq I_2$, then

$$u^*(t) = \begin{cases} b & \text{for } 0 \leq t \leq \hat{t} \ , \\ 0 & \text{for } \hat{t} < t \leq T \ . \end{cases}$$

2. If $\hat{t} > 0$, $0 < b < r$, and $I_1 < I(0) < I_3$, then

$$u^*(t) = \begin{cases} 0 & \text{for } 0 \leq t < \hat{\hat{t}} \ , \\ b & \text{for } \hat{\hat{t}} \leq t \leq \hat{t} \ , \\ 0 & \text{for } \hat{t} < t \leq T \ . \end{cases}$$

3. If $\hat{t} > 0$, $b \geq r$, and $0 \leq I(0) \leq I_3$, then

$$u^*(t) = \begin{cases} 0 & \text{for } 0 \leq t < \frac{I(0)}{r} \\ r & \text{for } \frac{I(0)}{r} \leq t \leq \hat{t} \\ 0 & \text{for } \hat{t} < t \leq T \ . \end{cases}$$

4. If $\hat{t} > 0$, $b \geq r$, and $I_2 < I(0) \leq 0$, then

$$u^*(t) = \begin{cases} b & \text{for } 0 \leq t \leq \frac{-I(0)}{[b-r]} \\ r & \text{for } \frac{-I(0)}{[b-r]} < t \leq \hat{t} \\ 0 & \text{for } \hat{t} < t \leq T \ . \end{cases}$$

5. Otherwise $u^*(t) = 0$ for all t in $[0,T]$.

Proof of the Optimal Policies--We have already shown that $u^*(t) = 0$ if $I(0) \geq rT$ or if $I(t) < 0$ and $\hat{t} < 0$ (that is, $T < \frac{k}{h_1}$) in our preliminary comments. These cases corresponded to $\frac{dH}{du} < 0$ over all t in $[0,T]$. We can add $I(t) < 0$ and $\hat{t} = 0$ to this list immediately because $\hat{t} = 0$ corresponds to $\frac{dH}{du} = 0$ at $t = 0$ and thus $\frac{dH}{du} < 0$ for all t in $(0,T]$.

Turning next to the case of $I(t) \leq 0$ and $\hat{t} > 0$ we realize that, because (17) applies, $\frac{dH}{du} > 0$ for t in $[0,\hat{t})$. In this time period $u^*(t) = b$ is the policy which will maximize $H(u)$. The period $(\hat{t},T]$ has $\frac{dH}{du} < 0$ in (17) and $u^*(t) = 0$. The switch from $u^*(t) = b$ to $u^*(t) = 0$ takes place at $\hat{t}$.

The conditions necessary for maintaining $I(t) \leq 0$ can be obtained from consideration of the following equation for $I(t)$ which results from the optimal policy just stated.

$$I(t) = \begin{cases} I(0) + [b-r]t & \text{for } 0 \leq t \leq \hat{t} \quad , \\ I(0) + b\hat{t} - rt & \text{for } \hat{t} < t \leq T \quad . \end{cases}$$

If $b < r$ then $I(t)$ decreases as t increases over $[0,T]$. Therefore, if $I(0) \leq 0$ then $I(t) \leq 0$ over $[0,T]$. If $b = r$ then $I(t) = I(0)$ as long as $u(t) = b$ and decreases as soon as $u(t) = 0$.

If $b > r$ it is possible to have $I(t) > 0$. Now if $b > r$ the maximum value of $I(t)$ occurs at $\hat{t}$. If $I(t)$ is to be nonpositive then we must have

$$I(\hat{t}) = I(0) + [b-r]\hat{t} \leq 0 \quad .$$

$I(\hat{t}) \leq 0$ will occur if

$$I(0) \leq [r-b]\hat{t} \equiv I_2 \quad ,$$

where $I_2 < 0$ because $b > r$.

In summary, we have just shown that

$$u^*(t) = \begin{cases} b & \text{for } 0 \leq t \leq \hat{t} \quad , \\ 0 & \text{for } \hat{t} < t \leq T \quad , \end{cases}$$

for all the conditions except $0 < I(0) \leq I_1$ for $0 < b < r$ of the first optimal policy.

For $I(0) > 0$ it is also possible to $\frac{dH}{du} > 0$ for some portion of $[0,T]$. We will concentrate on the range $0 < I(0) < rT$ because we have already shown that $u^*(t) = 0$ for $I(0) \geq rT$. If $u(t) = 0$ and $I(0)$ is in this range then $I(t) < 0$ for $t_1 < t \leq T$ where

$$t_1 = \frac{I(0)}{r} \quad . \tag{24}$$

The corresponding expression for $\frac{dH}{du}$ is

$$\frac{\partial H}{du} = \begin{cases} -k - h_2[t_1 - t] + h_1[T - t_1] & \text{for } 0 \leq t \leq t_1 \quad , \\ -k + h_1[T-t] & \text{for } t_1 \leq t \leq T \quad . \end{cases} \tag{25}$$

$\frac{dH}{du}$ has a slope of $(+h_2)$ in the region $0 \leq t \leq t_1$ and a slope of $(-h_1)$ in the region $t_1 \leq t \leq T$. A representative family of $\frac{dH}{du}$ curves for (25)

are shown in figure 1 and covers $I(0)$ values for $[0,rT]$. The curves designated by 1 and 5 correspond to $I(0) = 0$ and $I(0) = rT$, respectively. The optimality condition $\frac{dH}{du} = -k$ for $t = T$ is imposed on each curve of the figure.

Any curve between 1 and 2 suggests one switching; that is, $\frac{dH}{du}$ is positive for $t < \hat{t}$ and negative for $t > \hat{t}$. Any curve, such as 3 , between 2 and 4 suggests two switchings because $\frac{dH}{du}$ is negative, then positive, and then negative again. Any curve between 4 and 5 suggests no switching should take place.

It is immediately evident that conditions resulting in $\frac{dH}{du}$ curves between 4 and 5 can be added to the $u^*(t) = 0$ list. Curve 4 corresponds to $t_1 = \hat{t}$ which, when stated in terms of $I(0)$, results in

$$I(0) = r\hat{t} \equiv I_3$$

Therefore, if $I(0) \geq I_3$ then $u^*(t) = 0$ for all t in $[0,T]$.

Because $u(t) = 0$ is not optimal for the curves between 1 and 4 we will first investigate the behavior of $I(t)$ when we follow the switching patterns suggested by the curves. We will then re-evaluate the expressions for $\frac{dH}{du}$. This process may require several iterations before a pair of "matching" $I(t)$ and $\frac{dH}{du}$ curves can be obtained signalling that optimal $u(t)$ has been found.

One switching should be considered for conditions creating the curves between 1 and 2 ; the switching should occur at $\frac{dH}{du} = 0$. From figure 1 we know that $\frac{dH}{du} = 0$ only in the segment dominated by h_1 and that (17) applies. Therefore switching should occur at $\hat{t}$.

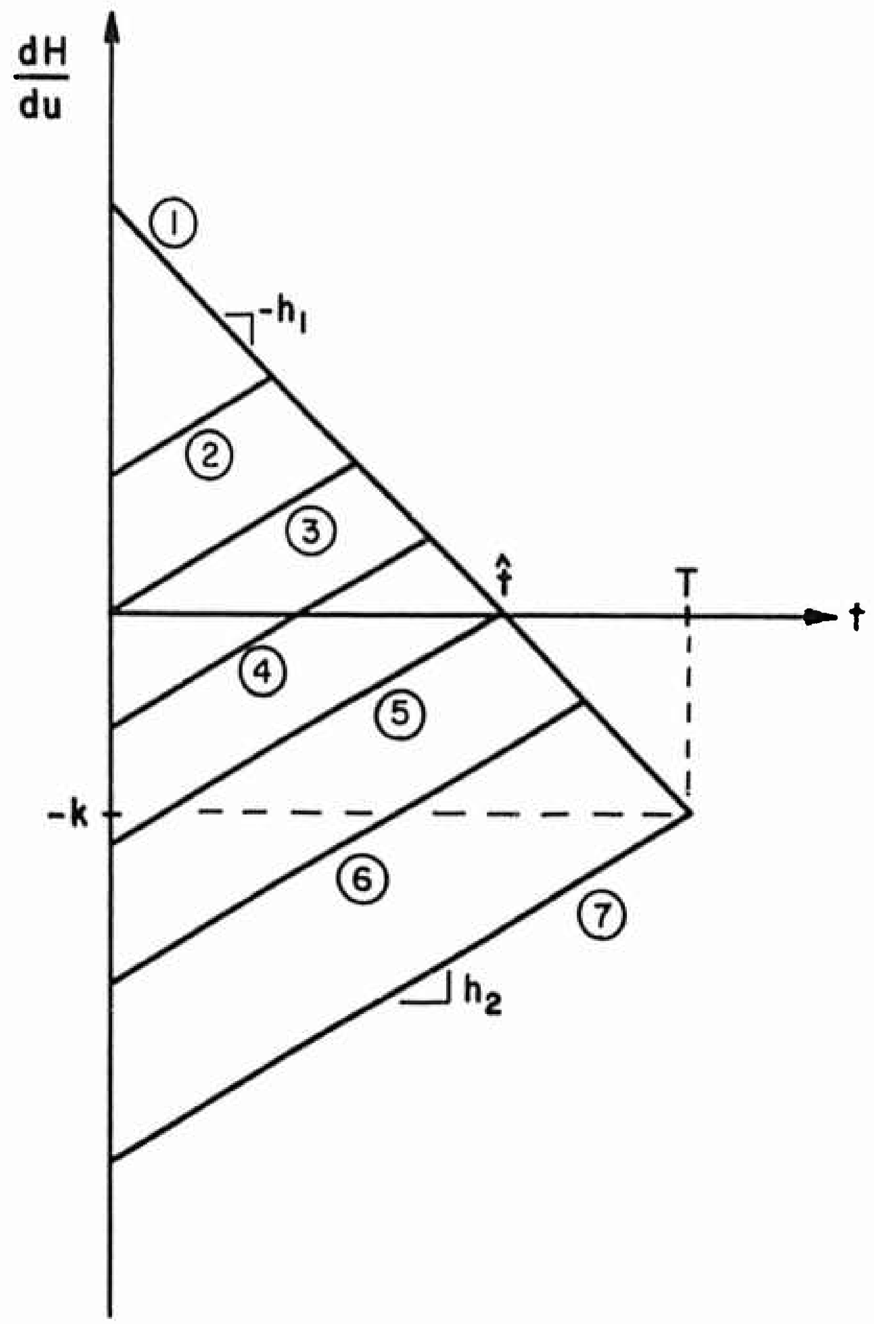

Figure 1. $\frac{dH}{du}$ curves for equation (25).

If we let

$$u(t) = \begin{cases} b & \text{for } 0 \le t \le \hat{t} \ , \\ 0 & \text{for } \hat{t} < t \le T \ , \end{cases}$$

then, from (1),

$$I(t) = \begin{cases} I(0) + [b-r]t & \text{for } 0 \le t \le \hat{t} \ , \\ I(0) + b\hat{t} - rt & \text{for } \hat{t} < t \le T \ . \end{cases} \tag{26}$$

A family of $I(t)$ curves for (26) is shown in figure 2. Curve 1 is representative of a case when $b > r$. Curve 2 results from $b = r$. Curves 3, 4, and 5 correspond to $b < r$. Curve 5 has $b = 0$ and is the limiting curve for the $b < r$ cases.

Figure 3 shows the curves of $\frac{dH}{du}$ corresponding to $I(t)$ curves of figure 2. The same curve numbers have been used in both figures to show the correspondence. Curves between 1 and 3 have $\frac{dH}{du} < 0$ over $[0,T]$ and suggest that no switching is optimal. Curves between 3 and 5 suggest that at least one switching is optimal.

For any curves between 4 and 5 we see that $\frac{dH}{du}$ has the same behavior as in figure 1 for curves between 1 and 2. Therefore,

$$u^*(t) = \begin{cases} b & \text{for } 0 \le t \le \hat{t} \ , \\ 0 & \text{for } \hat{t} < t \le T \ , \end{cases}$$

for the conditions leading to these $\frac{dH}{du}$ curves.

The first condition is that $0 < b < r$. The second is that $\frac{dH}{du} \ge 0$ at $t = 0$. Translating this into a relationship involving $I(0)$ we have

$$I(0) \le \frac{[r-b]}{h_1 + h_2} [h_1 T - k] \equiv I_1 \ .$$

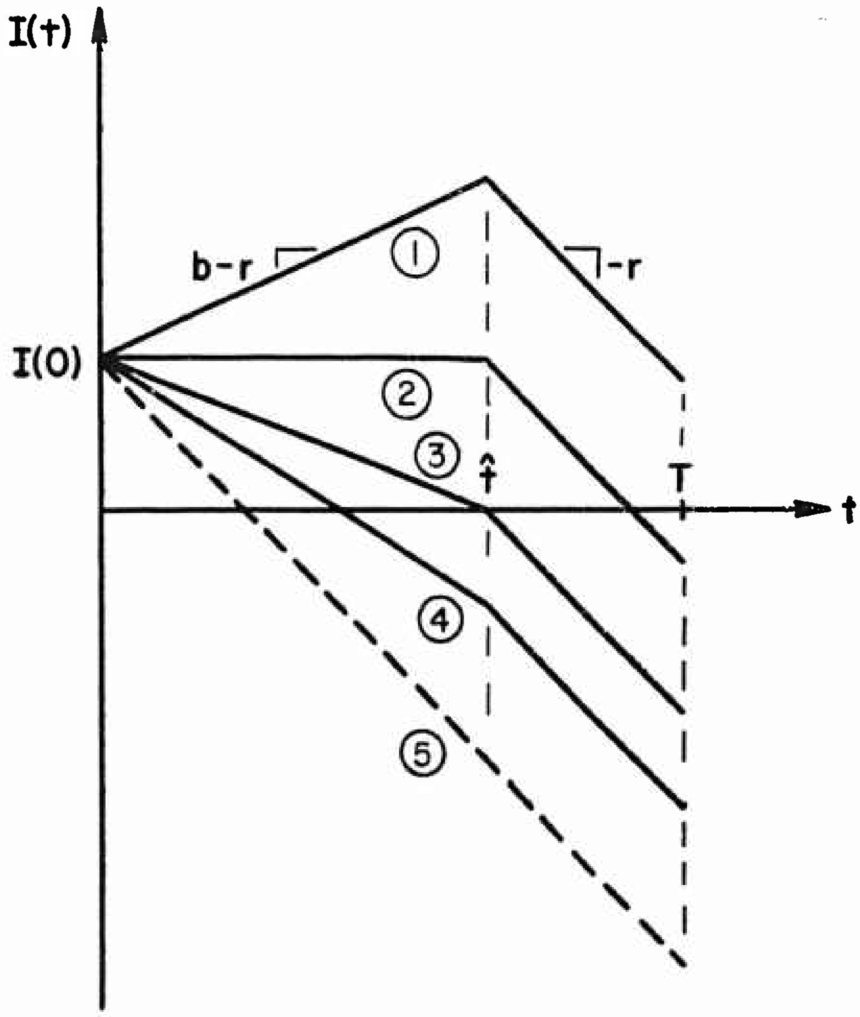

Figure 2. Curves of I(t) for equation (26).

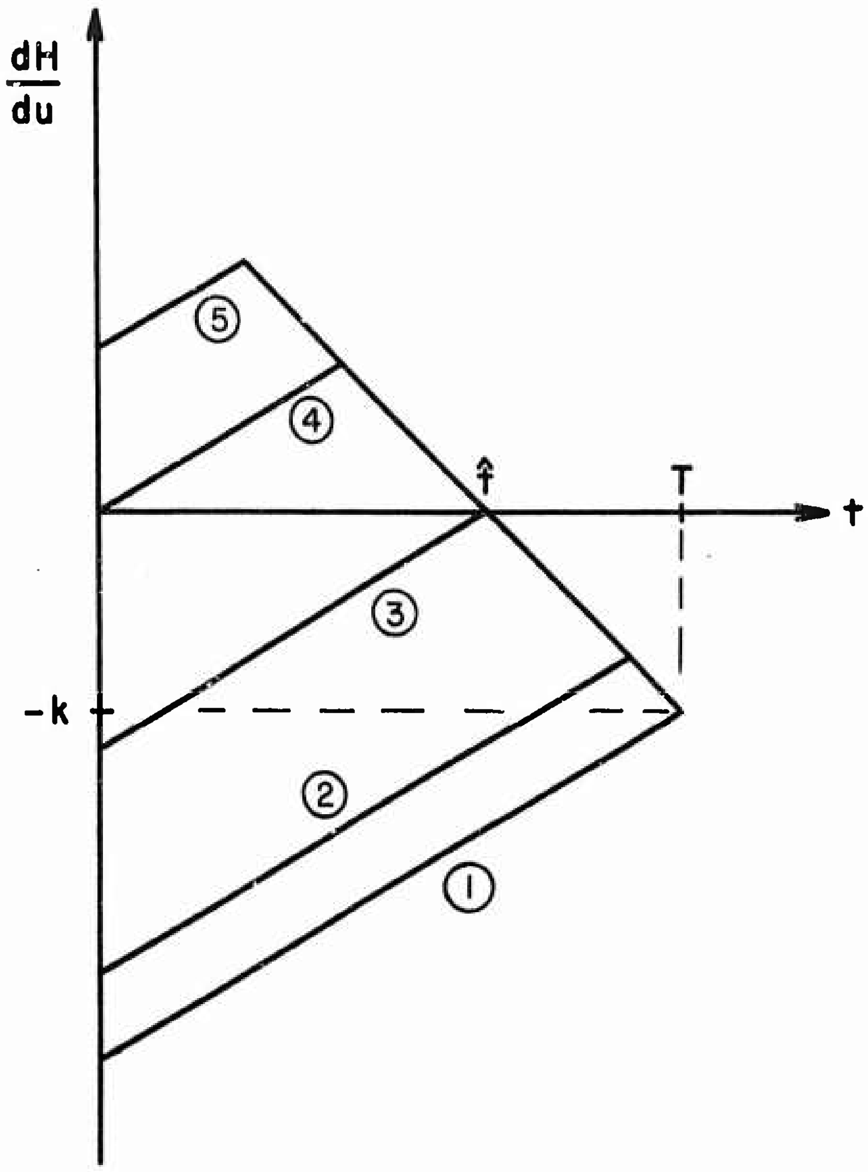

Figure 3. $\frac{dH}{du}$ curves associated with the I(t) curves of figure 2.

This analysis has been based on the assumption that $I(0) > 0$, therefore the proof of the first optimal policy is complete.

We consider next the curves between 3 and 4 . Figure 2 indicates that $0 < b < r$ and figure 3 shows that two switchings may be optimal; the first at some time $\hat{\hat{t}}$ prior to $\hat{t}$ and the second at $\hat{t}$. The following policy is therefore suggested.

$$u(t) = \begin{cases} 0 & \text{for} \quad 0 \le t < \hat{\hat{t}} \ , \\ b & \text{for} \quad \hat{\hat{t}} \le t \le \hat{t} \ , \\ 0 & \text{for} \quad \hat{t} < t \le T \ . \end{cases} \tag{27}$$

The corresponding expression for $I(t)$ would then be

$$I(t) = \begin{cases} I(0) - rt & \text{for} \quad 0 \le t < \hat{\hat{t}} \\ I(0) + b[t-\hat{\hat{t}}] - rt & \text{for} \quad \hat{\hat{t}} \le t \le \hat{t} \\ I(0) + b[\hat{t}-\hat{\hat{t}}] - rt & \text{for} \quad \hat{t} < t \le T \ . \end{cases} \tag{28}$$

Figure 4 illustrates the family of curves associated with (28) and figure 5 presents the corresponding $\frac{dH}{du}$ curves. From these two figures we see that curves 2 and 3 will have $u^*(t)$ given by (27). To find the value of $\hat{\hat{t}}$ we set $\frac{dH}{du} = 0$ in equation (19) where now $I(t) = 0$ at

$$t_1 = \hat{\hat{t}} + \frac{I(0) - r\hat{\hat{t}}}{b - r} \ .$$

We will get the expression given by (23) and $\hat{\hat{t}} > 0$ if

$$I(0) > \frac{h_1[r-b]\hat{t}}{h_1 + h_2} \equiv I_1 \ .$$

Similarly, $\hat{\hat{t}} < \hat{t}$ if

$$I(0) < r\hat{t} \equiv I_3 \ .$$

When $I(0) = I_1$ we get $\hat{\hat{t}} = 0$ and the optimal policy given by

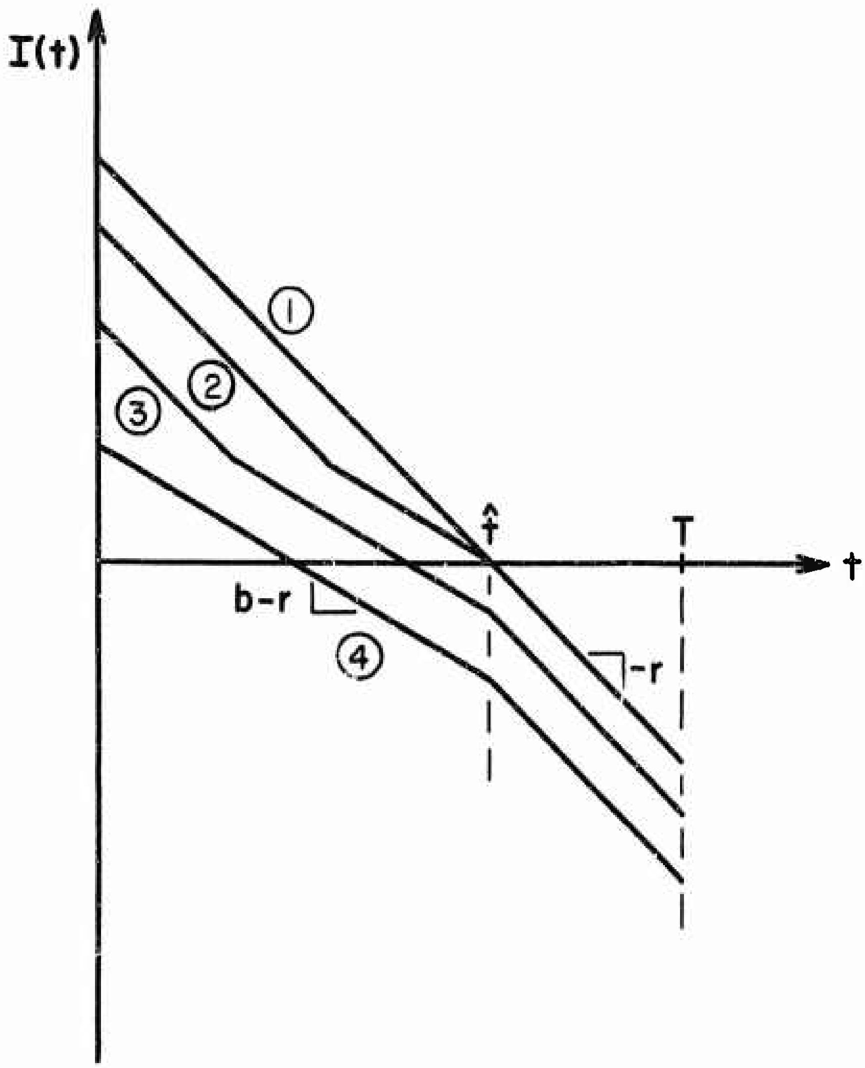

Figure 4. Curves of I(t) for equation (28).

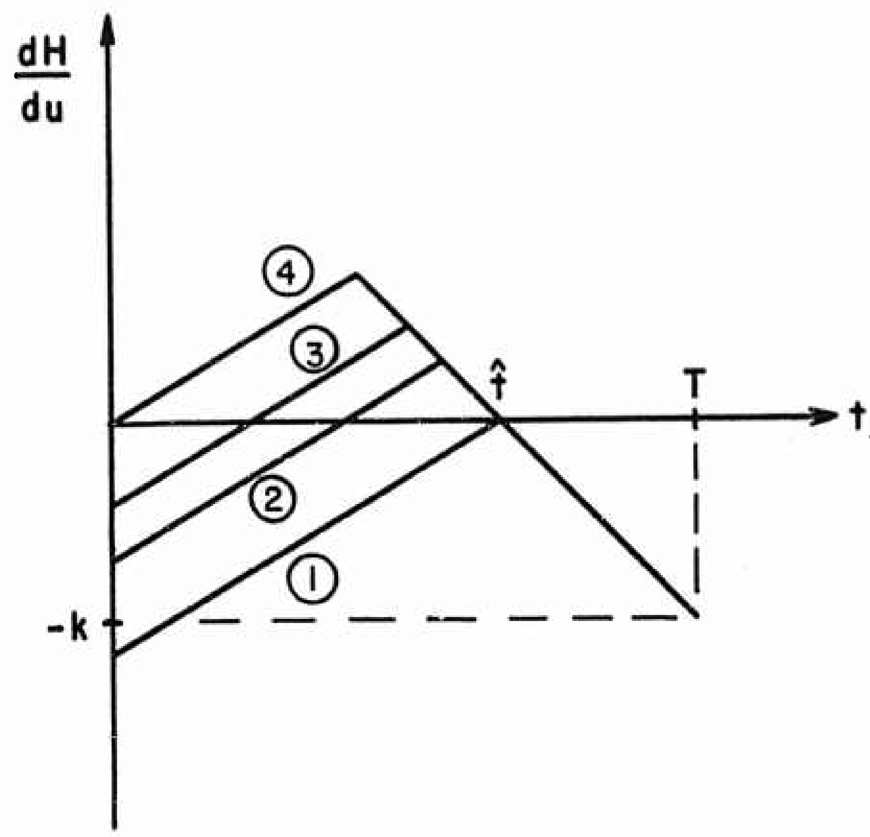

Figure 5. Curves of $\frac{dH}{du}$ associated with the I(t) curves of figure 4.

(27) reduces to that derived above for the case of $I(0) \leq I_1$ and $I(t)$ corresponds to curve 4 of figure 4. If $\hat{\hat{t}} = \hat{t}$ then (27) reduces to $u^*(t) = 0$ and $I(t)$ corresponds to curve 1 of figure 4. We have already shown that $u^*(t) = 0$ if $I(0) \geq I_3$ regardless of the relationship of b to r. In summary, when $\hat{t} > 0$, $b < r$, and $I_1 < I(0) \leq I_3$ then $u^*(t)$ is given by (27) and the second optimal policy is confirmed.

To complete the analysis of cases for $I(0) \geq 0$ we next consider the case of $b \geq r$. When $b = r$ the bound I_1 is zero while I_3 does not change so $0 \leq I(0) \leq I_3$. Equation (23) reduces to

$$\hat{\hat{t}} = \frac{I(0)}{r} .$$

We realize that $\hat{\hat{t}}$ occurs simultaneously with t_1 as given by (24) and hence $I(t) = 0$ between $\hat{\hat{t}}$ and $\hat{t}$. Equation (31) shows that $\frac{dH}{du}$ is constant over this interval because $\frac{dH}{du} = 0$ for $I(t) = 0$ was assumed.

$$\frac{dH}{du} = \begin{cases} -k - h_2[\hat{\hat{t}}-t] + h_1[T-\hat{t}] & \text{for } 0 \leq t \leq \hat{\hat{t}} \\ -k + h_1[T-\hat{t}] & \text{for } \hat{\hat{t}} \leq t \leq \hat{t} \\ -k + h_1[T-t] & \text{for } \hat{t} \leq t \leq T . \end{cases} \quad (31)$$

From the definition of $\hat{t}$ we also realize that this constant value of $\frac{dH}{du}$ must be zero. Therefore $u^*(t)$ can take on any value between 0 and b. However, $I(t) = 0$ for t in $[\hat{\hat{t}},\hat{t}]$ only if $u^*(t) = b$.

This argument can be easily extended to the case of $b > r$. The optimal policy should take the following form, however;

$$u^*(t) = \begin{cases} 0 & \text{for } 0 \leq t < \hat{\hat{t}} \\ r & \text{for } \hat{\hat{t}} \leq t \leq \hat{t} \\ 0 & \text{for } \hat{t} < t \leq T . \end{cases} \quad (32)$$

The corresponding expression for $\frac{dH}{du}$ is again given by (31) and since $\frac{dH}{du} = 0$ over $\hat{\hat{t}} \leq t \leq \hat{t}$ we can assign any value of u as optimal between 0 and b. The only value of u which will maintain $I(t) = 0$ over $[\hat{\hat{t}}, \hat{t}]$ is now $u(t) = r$. Figure 6 illustrates the resulting curves of $I(t)$ and $\frac{dH}{du}$. The third optimal policy has been proved.

These arguments can also be used for the case of $b > r$ and $I_2 < I(0) \leq 0$. We recall that $u(t) = b$ for $[0, \hat{t}]$ and zero outerwise would result in $I(\hat{t}) > 0$. Figure 7 contains possible $I(t)$ curves. The corresponding $\frac{dH}{du}$ curves are shown in figure 8.

The curves 3 in the two figures correspond to the case of $I(0) = I_2$ and the first optimal policy applies (equation (32) would have $\hat{\hat{t}} = 0$). The curves 1 and 2 from figure 7 do not allow $\frac{dH}{du} = -k$ for $t = T$ and hence $u(t)$ given by (32) is not optimal. However, if we assume the following form for $u(t)$ we can satisfy this condition.

$$u(t) = \begin{cases} b & \text{for } 0 \leq t \leq \hat{\hat{t}} , \\ r & \text{for } \hat{\hat{t}} \leq t \leq \hat{t} , \\ 0 & \text{for } \hat{t} \leq t \leq T , \end{cases} \tag{33}$$

where now

$$\hat{\hat{t}} = \frac{I(0)}{[b-r]} .$$

The resulting equation for $\frac{dH}{du}$ is:

$$\frac{dH}{du} = \begin{cases} -k + h_1[\hat{\hat{t}}-t] + h_1[T=\hat{t}] & \text{for } 0 \leq t \leq \hat{\hat{t}} \\ -k + h_1[T-\hat{t}] & \text{for } \hat{\hat{t}} \leq t \leq \hat{t} \\ -k + h_1[T-t] & \text{for } \hat{t} \leq t \leq T . \end{cases}$$

We realize, as we did for (31), that $\frac{dH}{du} = 0$ for $\hat{\hat{t}} \leq t \leq \hat{t}$ and

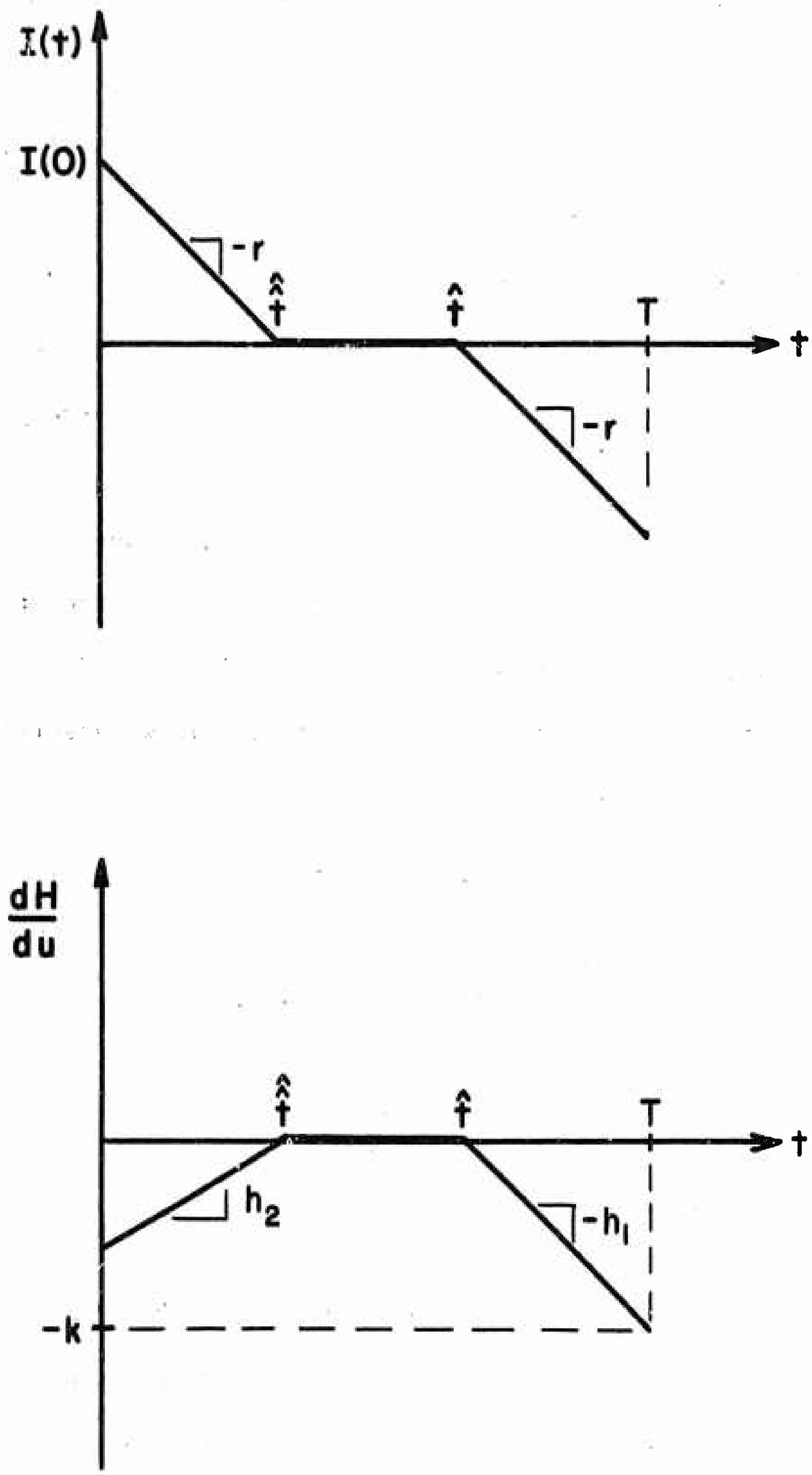

Figure 6. I(t) and $\frac{dH}{du}$ curves for $u^*(t)$ given by equation (32).

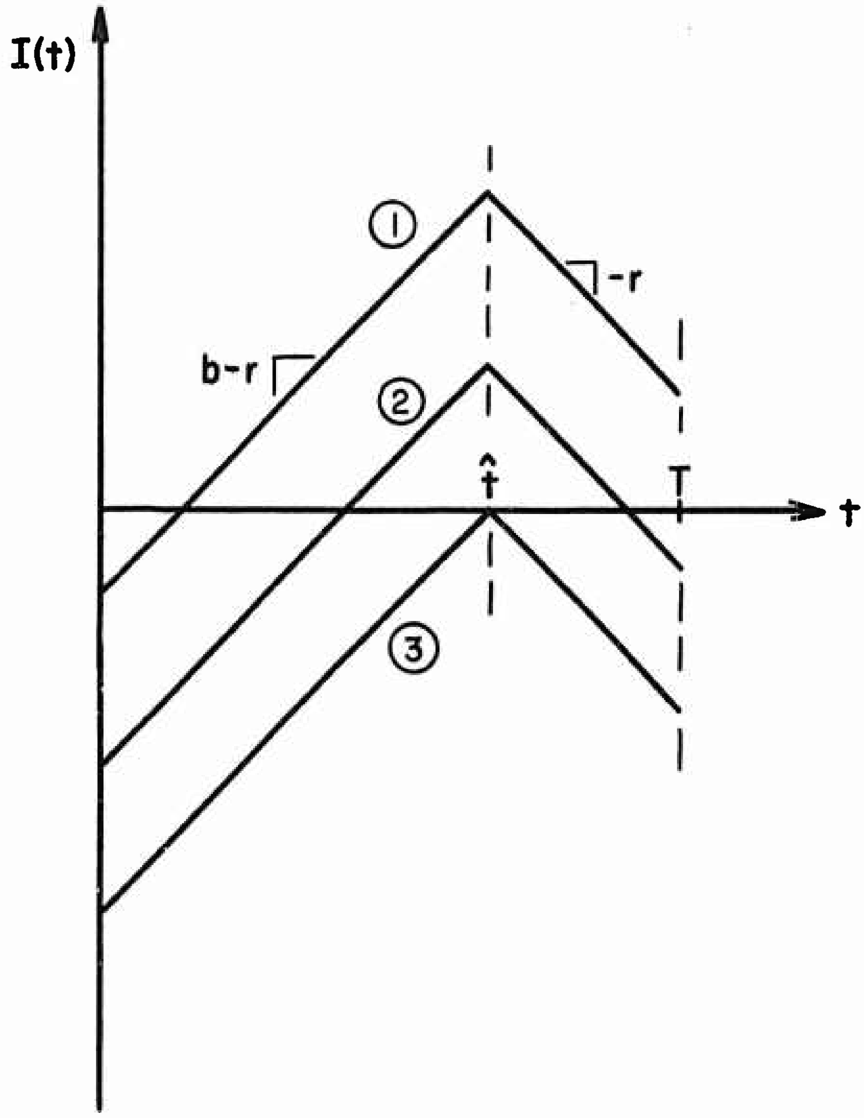

Figure 7. Possible $I(t)$ curves for $b > r$, $I(0) < 0$, $u(t) = b$ over $[0,\hat{t}]$.

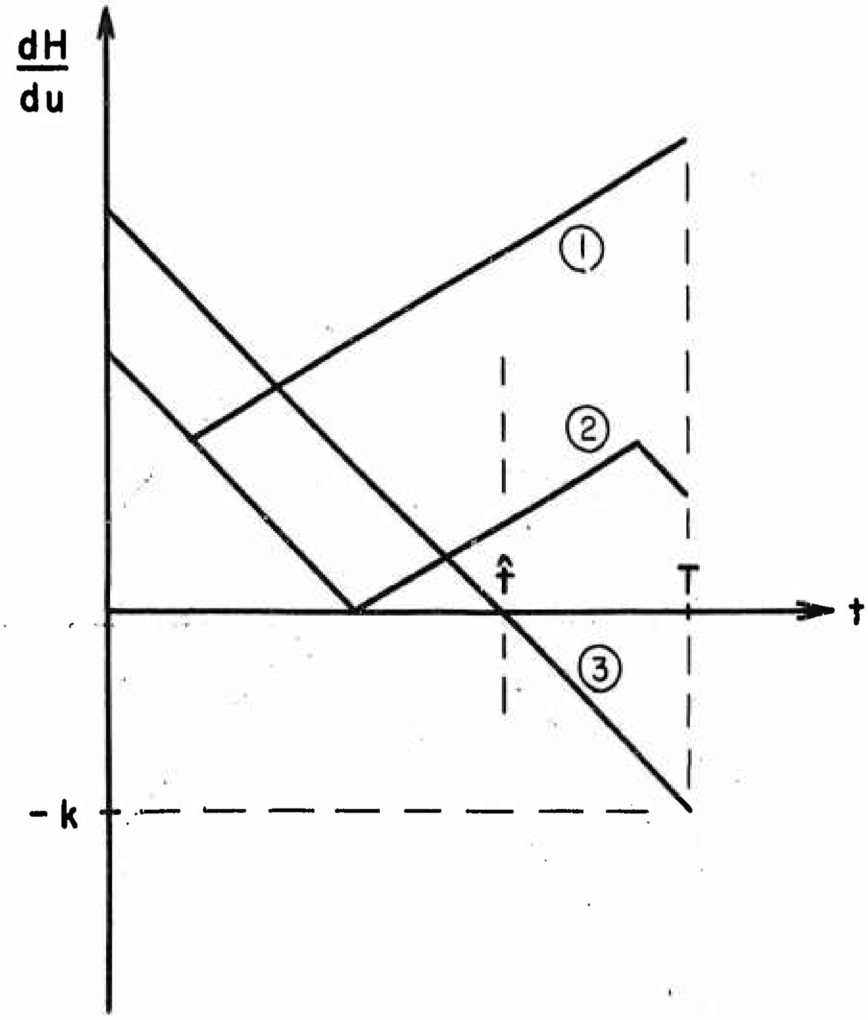

Figure 8. Curves of $\frac{dH}{du}$ associated with the I(t) curves of figure 7.

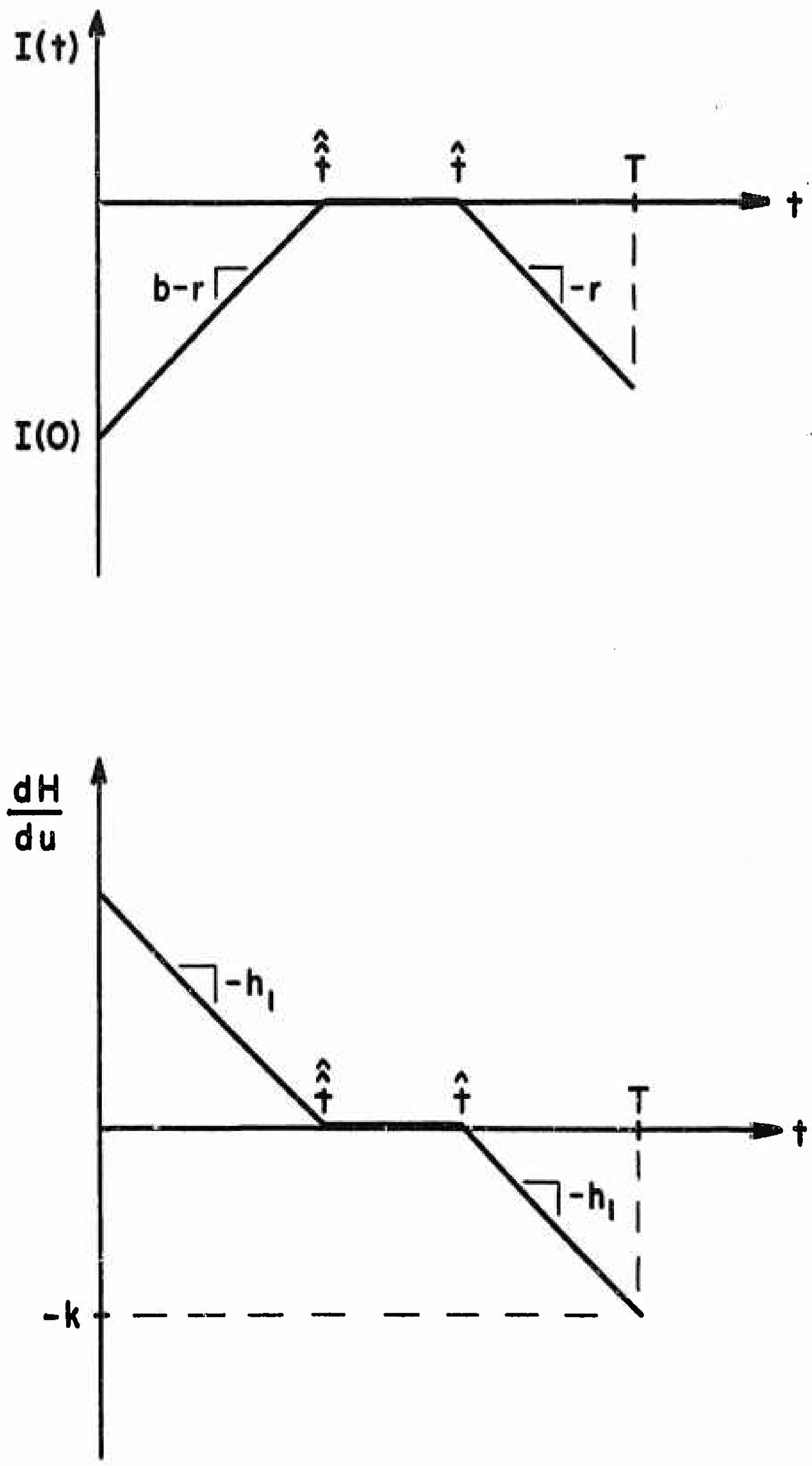

Figure 9. Curves of I(t) and $\frac{dH}{du}$ for $u^*(t)$ given by (33).

that we should select $u(t) = r$ to maintain $I(t) = 0$ over this interval. Therefore (33) gives the optimal policy when $I_2 < I(0) \leq 0$. The curves of $I(t)$ and $\frac{dH}{du}$ are shown in figure 9. The proof of the fourth optimal policy is complete.

Comments--When stockouts are allowed we see that the analysis becomes immediately more complex because the $I(t) \geq 0$ constraint has been removed and therefore the region of feasible $u(t)$ has expanded to always include $u(t) = 0$. The assumption of an upper bound on u and specific forms for the elements of the cost function appear necessary before an analysis can proceed.

The linear problem presented in the preceding section was chosen for the illustration of the stockouts-allowed analysis primarily because the cost functions are typical of many deterministic models appearing in the literature. Future analyses will investigate the influence of several different nonlinear cost forms. It should be evident from the preceding analysis that while the use of the Maximum Principle allows a general order policy to be postulated, it requires that careful consideration be given to the form of the objective function. In contrast to the no-stockouts model, the nature of the form selected will strongly influence the nature of the optimal order policy.

The iterative procedure of postulating $u(t)$, evaluating $I(t)$ and $\frac{dH}{du}$, and re-postulating $u(t)$ until finally a "matched set" of expressions for $I(t)$ and $\frac{dH}{du}$ is obtained seems to hold considerable promise for problems having nonlinear cost functions. The virtue in obtaining the "matched set" is that then both the necessary and sufficient conditions for optimality have been obtained.

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13. ABSTRACT

The use of Pontryagin's Maximum Principle in analyses of deterministic inventory models requires less restrictive assumptions about the nature of order policies than can be allowed when only the differential calculus is used. Two analyses are presented to illustrate this fact. The first involves a periodic review model with no shortages allowed. The production and holding cost functions were kept as general as possible. The second analysis involves a periodic review model with shortages allowed and linear production and inventory costs. As would be expected, the optimal order policies obtained are more general than those obtained in the past.

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